Phase Representation of Analytic Functions*

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The phase representation which expresses an analytic function essentially in terms of its phase (not imaginary part) along the cuts is discussed. In particular, the precise conditions under which this phase representation is valid and also the asymptotic behavior of the phase representation are studied in detail. It is proved that the asymptotic behavior is essentially the same at infinity in all directions. The derivation of the high-energy behavior of scattering amplitudes and the N/D representation of the partial-wave amplitude are discussed as applications of the phase representation. Finally, the phase representation is used in determining the total numbers of zeros of the forward pion-nucleon scattering amplitudes. It is found that the charge nonexchange amplitude has either 2 or 4 zeros, depending upon the signs of the S-wave scattering lengths, while the charge exchange amplitude has 11 zeros.

I. INTRODUCTION AND SUMMARY

THE dispersion relation for a real analytic function has the general form

$$f(z)=\frac{1}{\pi}\int_{\text{cuts}}\frac{\operatorname{Im}f(x+i\epsilon)}{x-z}dx+\text{pole terms},\qquad(1)$$

where the integral is along the cuts on the real axis and ϵ is an infinitesimal positive number. The dispersion relation (1) expresses $f(z)$ essentially in terms of its imaginary part along the cuts.

The purpose of this paper is to discuss the properties and the uses of an alternative representation of $f(z)$, which involves its (real) phase along the cuts. The phase $\delta(x)$ is defined by

$$f(x+i\epsilon)=\pm|f(x+i\epsilon)|e^{i\delta(x)}.\qquad(2)$$

This phase representation is expressed as

$$f(z)=\frac{P_1(z)}{P_2(z)}\exp\left(\frac{z}{\pi}\int_{\text{cuts}}\frac{\delta(x)dx}{x(x-z)}\right),\qquad(3)$$

where the integral is also along the cuts and $P_1(z)$ and $P_2(z)$ are finite polynomials in z. The representation (3) was first worked out by Omnes and Muskhelishvili[1] and used by many authors[2] in discussing partial-wave amplitudes. The present authors[3] also gave a proof of (3), together with a useful application. It was proved[3] that (3) is valid under the conditions (a), (b), (c), and (d) listed in Sec. 2 of this paper. The extra conditions in reference 3 are shown, in fact, to be unnecessary in Sec. 2. To our knowledge, no one[1,2] has observed the precise conditions (a), (b), (c), and (d) for (3) to be valid, or recognized that the phase representation has the simple asymptotic form given by (9). This asymptotic behavior is used in most of the applications of (3) presented in this paper.

The phase $\delta(x)$ is not uniquely defined by (2), especially at those points where $f(x)$ vanishes. The phase representation is valid, independently of the specific definition of $\delta(x)$. However, it appears most convenient to define $\delta(x)$ as follows: $\delta(x)$ is zero on the real axis where no cuts occur and has no discontinuities greater than or equal to π in magnitude. The $\pm$ sign in (2) is then uniquely defined. We assume the above definition throughout this paper.

We show in Sec. 2 that the discontinuities in $\delta(x)$ are associated with branch points of the type given by (4) and/or (5) and do not cause any zeros or poles in the exponential function in (3) except at infinity. Thus, the polynomials $P_1(z)$ and $P_2(z)$ in (3) account for all the zeros and poles of $f(z)$ except the one at infinity. The polynomial $P_2(z)$ is finite because the number of poles of $f(z)$ is finite according to the condition (a). The polynomial $P_1(z)$ is also finite because the exponential function of (3) is bounded by a finite polynomial at infinity due to the condition (d). This means that $f(z)$ has only a finite number of zeros under the conditions (a), (b), (c), and (d) of Sec. 2.

We prove also in Sec. 2 that the phase representation is valid under the conditions (a), (b), (c), and that $f(z)$ has only a finite number of zeros. In this case the condition (d) is implied by the finite number of zeros of $f(z)$. Thus, the condition (d), is essentially equivalent to $f(z)$ having a finite number of zeros.

In Sec. 3, the phase representation is used to prove the theorem[4] which states that the limit of $f(z)$ at infinity is essentially the same in all directions in the z plane when $f(z)$ satisfies the conditions (a), (b), (c), and (d).

In Sec. 4, we refer to the derivation of the high-energy behavior of the scattering amplitude by means of the phase representation, which was the subject matter of our previous note.[3]

In Sec. 5, we briefly discuss the phase representation

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[1] See, for example, the review by J. D. Jackson, in *Dispersion Relations*, edited by G. R. Screaton (Interscience Publishers, Inc., New York, 1961), p. 54.

[2] Extensive use of the phase representation has been made recently by G. Frye and R. L. Warnock, Phys. Rev. **130**, (1963), which also gives references to virtually all previous work.

[3] M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962).

[4] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961). The proof in Sec. 3 was motivated by and was completed in collaboration with M. Froissart.

in the case of partial-wave amplitudes.[2] The details regarding the N/D method and the inverse method have already been published.[5]

In Sec. 6, the phase representation is used to locate all the zeros of the forward pion-nucleon scattering amplitudes in the entire energy plane. The number of zeros of the charge nonexchange amplitude is either 2 when $a_1+2a_3<0$, or 4 when $a_1+2a_3\geq 0$, where a_1 and a_3 are the S-wave scattering lengths in the channels with the total isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively. In both cases, two zeros occur on the imaginary axis and the other two, when they appear, on the gap along the real axis. The charge exchange amplitude is found to have 11 zeros, one of which occurs at the origin because of the odd symmetry.

We conclude this section by making some remarks on the connection between the dispersion relation (1) and the phase representation (3). First, we note that any function which has the phase representation with finite $\delta(\pm\infty)$ satisfies the dispersion relation except for a certain number of subtractions. Moreover, since the phase representation satisfies the theorem of reference 4, the number of necessary subtractions is uniquely determined by the behavior of $f(z)$ as z approaches, say, the positive real infinity. This is, in fact, how we usually determine the number of subtractions. This general practice, therefore, becomes legitimate when we assume the phase representation.

However, the converse of the statement at the beginning of the previous paragraph is not necessarily correct, because the phase does not have to be bounded at infinity when $f(z)$ approaches zero at infinity.[6] However, if we add a real constant to $f(z)$, the modified $f(z)$ will have a bounded phase and consequently the phase representation.

The major differences between the dispersion relation (1) and the phase representation (3) are as follows: First, the subtraction is hidden in the phase representation in the sense that (3) does not depend in form on how $f(z)$ behaves at infinity, while the subtraction affects the expression (1). Secondly, the zeros of $f(z)$ are hidden in the dispersion relation, while they are explicit in $P_1(z)$ of the phase representation (3). These differences are responsible for the fact that some of the results which follow from the phase representation would have been very difficult to derive using only the dispersion relation. It is likely that these two representations are generally complementary in the sense that one is likely to be practically useless when the other turns out to be very useful. It seems, however, worthwhile to try to exploit both representations to the maximum extent.

2. PHASE REPRESENTATION

We show in this section that the phase representation (3) is valid under the following conditions:

(a) $f(z)$ is analytic in z everywhere except for cuts which occur on the real axis and a finite number of poles,

(b) $f(z)$ is real in the sense that $f^*(z)=f(z^*)$,

(c) $f(z)$ is bounded at $|z|=\infty$ by a finite polynominal in z,

(d) $\delta(x)$ has finite limits $\delta(\pm\infty)$ as $x\rightarrow\pm\infty$.

The proof of (3) in reference 3 applies to the case when $f(z)$ is a scattering amplitude, and therefore satisfies, besides those conditions listed above, the additional ones that $\delta(x)$ is continuous in x and $\delta(x)=-\delta(-x)$ because of crossing symmetry.

We show in this section that these extra conditions are, in fact, unnecessary. For this, we show that the exponential function in (3), even without these extra conditions, continues to have no zeros or poles except at infinity and also remains bounded by a finite polynomial at infinity. It is then straightforward to see that the proof of reference 3 holds without change.

The fact that the exponential function in (3) has no poles or zeros except at infinity is due to our definition of the phase $\delta(x)$. According to our definition, the discontinuities in $\delta(x)$ are smaller in magnitude than π. Suppose $\delta(x)$ has discontinuities of a_i at $x=x_i$, $i=0, 1, 2, \cdots$, along the cut which extends from x_0 to $+\infty$, and of b_j at $x=y_j$, $j=0, 1, 2, \cdots$, along the cut which extends from y_0 to $-\infty$. We may assume without loss of generality that $f(z)$ has only the cuts just mentioned. Let us define a continuous phase $\delta'(x)$ which is to remain zero on the gap, by shifting the phase by constants a_i, b_j at $x=x_i$, y_j, respectively. The exponential function in (3) can then be factored into the exponential function with the continuous phase $\delta'(x)$ in the integrands and factors of the type

$$\exp\left(\frac{z}{\pi}\int_{x_i}^{\infty}\frac{a_i dx}{x(x-z)}\right)=\left(\frac{|x_i|}{x_i-z}\right)^{a_i/\pi}, \tag{4}$$

and/or

$$\exp\left(\frac{z}{\pi}\int_{-\infty}^{y_j}\frac{-b_j dx}{x(x-z)}\right)=\left(\frac{z-y_j}{|y_j|}\right)^{-b_j/\pi}, \tag{5}$$

depending upon the locations of the discontinuities. None of these factors have zeros or poles except at infinity as long as the discontinuities a_i and b_j are smaller than π in magnitude.

We then split the integrals of the exponential function in (3) with the continuous phase $\delta'(x)$ in the integrands

[5] M. Sugawara and A. Kanazawa, Phys. Rev. **126**, 2251 (1962).

[6] Consider the function $f(z)=[\exp(1+i\sqrt{z})-1]/(1+i\sqrt{z})$, with a cut from 0 to $+\infty$ along the real axis. This function satisfies the conditions (a), (b), (c), and approaches zero at infinity in all directions, therefore satisfying a no-subtraction dispersion relation. However, the phase of $f(z)$ along the cut is not bounded at infinity; and therefore, $f(z)$ has no phase representation. Also $f(z)$ has an infinite number of zeros, as it should according to our argument. The above example is due to M. A. Ruderman (private communication).

as follows:

$$\frac{\delta'(+\infty)}{\pi}\ln\left(\frac{|x_0|}{x_0-z}\right)+\frac{\delta'(-\infty)}{\pi}\ln\left(\frac{z-y_0}{|y_0|}\right)$$
$$+\frac{1}{\pi}\int_{x_0}^{\infty}\left(\frac{1}{x-z}-\frac{1}{x+z}\right)[\delta'(x)-\delta'(+\infty)]dx$$
$$-\frac{1}{\pi}\int_{x_0}^{\infty}\left(\frac{1}{x}-\frac{1}{x+z}\right)[\delta'(x)-\delta'(+\infty)]dx$$
$$+\frac{1}{\pi}\int_{-y_0}^{\infty}\left(\frac{1}{x}-\frac{1}{x+z}\right)[\delta'(-x)-\delta'(-\infty)]dx. \quad (6)$$

The logarithmic terms in (6) are the integrals with $\delta'(x)$ replaced by $\delta'(+\infty)$ and $\delta'(-\infty)$, respectively. We now combine the last two integrals into a single infinite integral and some finite integrals, by shifting the lower integration limits. These finite integrals approach finite numbers asymptotically, and the rest of (6) is then, precisely of the form of Eq. (2) of reference 3, even in the absence of the symmetry of $\delta(x)$. Therefore, the arguments of reference 3 apply to (6). In particular, only when the integral

$$\int^{\infty}\{[\delta'(x)-\delta'(+\infty)]-[\delta'(-x)-\delta'(-\infty)]dx/x \quad (7)$$

converges, do the integrals in (6) remain finite as $|z|\to\infty$. The exponential function in (3) then approaches, as $|z|\to\infty$, a simple power form

$$(-z)^{-\delta(+\infty)/\pi}(z)^{\delta(-\infty)/\pi} \quad (8)$$

except for a real constant factor. In obtaining (8) we have used the identities that $\delta(+\infty)=\delta'(+\infty)+\sum_i a_i$ and $\delta(-\infty)=\delta'(-\infty)-\sum_j b_j$. If the integral (7) does not converge, (8) is modified by a factor which behaves, as $|z|\to\infty$, only logarithmically. Therefore, we see that the exponential function in (3) is bounded by a finite polynomial at infinity as long as $\delta(\pm\infty)$ are finite.[7] We note that the discontinuities are completely irrelevant in the asymptotic form (8).

The asymptotic behavior of the phase representation (3) is summarized as follows: As $|z|\to\infty$, (3) approaches a simple power form

$$z^{n-m}(-z)^{-\delta(+\infty)/\pi}(z)^{\delta(-\infty)/\pi}, \quad (9)$$

except for a real constant factor, only when the integral (7) converges. The integers n and m in (9) are the orders of polynomials $P_1(z)$ and $P_2(z)$ in (3) and, therefore, the total numbers of zeros and poles of $f(z)$, respectively. If the integral (7) diverges, (9) must be modified by a factor which depends only logarithmically on z as $|z|\to\infty$.

We prove in the following that the phase representation (3) is valid under the conditions (a), (b), (c), and that $f(z)$ has a finite number of zeros. Now, since $f(z)$ has finite numbers of zeros and poles, it can be written as $f(z)=[P_1(z)/P_2(z)]Q(z)$, where $P_1(z)$ and $P_2(z)$ are finite polynomials and $Q(z)$ has no zeros or poles except at infinity. We then observe that $\ln Q(z)$ is analytic in z everywhere except for the cuts of $f(z)$ and is bounded by a logarithmic function at infinity. Therefore, $\ln Q(z)$ satisfies a once-subtracted dispersion relation, the spectral function of which is

$$(1/2i)[\ln Q(x+i\epsilon)-\ln Q(x-i\epsilon)]$$
$$=\frac{1}{2i}\ln\left(\frac{Q(x+i\epsilon)}{Q(x-i\epsilon)}\right)=\frac{1}{2i}\ln\left(\frac{f(x+i\epsilon)}{f(x-i\epsilon)}\right)=\delta(x), \quad (10)$$

according to our definition of the phase $\delta(x)$. This dispersion relation for $\ln Q(z)$ now represents the exponential function in (3) except for a trivial constant factor. Therefore, $f(z)$ satisfies the phase representation (3). When $f(z)$ has a finite number of zeros, the exponential function in (3) must be bounded by a finite polynomial at infinity. This implies[7] that $\delta(\pm\infty)$ are finite. Therefore, the condition (d) is essentially equivalent to $f(z)$ having a finite number of zeros.

3. THEOREM CONCERNING THE LIMIT AT INFINITY

It was proved[4] that a function $f(z)$ which satisfies the conditions (a), (b), and (c) of Sec. 2 behaves in a simple manner at infinity. If $f(z)$ has finite limits $f(+\infty\pm\epsilon i)$ as $z\to+\infty\pm i\epsilon$ along the cut extending to $+\infty$ and $f(z)$ *approaches definite limits*[8] (*not necessarily finite*) as $z\to-\infty\pm i\epsilon$ when there is the second cut extending to $-\infty$, then $f(z)$ approaches at infinity either $f(+\infty+i\epsilon)$ or $f(+\infty-i\epsilon)$ in all directions in the upper or lower half-plane, respectively.

The condition that $f(z)$ approaches definite limits[8] at the end of the second cut is due to the fact that $f(-\infty\pm i\epsilon)$ are treated throughout the proof of reference 4 as definite complex numbers. In order for $f(-\infty\pm i\epsilon)$ to be definite complex numbers, it is necessary for the phase $\delta(x)$ to have a finite limit $\delta(-\infty)$ as $x\to-\infty$, unless $f(-\infty\pm i\epsilon)$ happen to vanish. We can avoid the case when $f(-\infty\pm i\epsilon)$ and also $f(+\infty\pm i\epsilon)$ happen to vanish, by adding to $f(z)$ an arbitrary real

[7] It may appear offhand that only $\delta(+\infty)-\delta(-\infty)$ need be finite, but not necessarily both $\delta(\pm\infty)$. However, we assume in (6) that both $\delta(\pm\infty)$ are finite. Otherwise, we do not know how we could prove the boundedness of the exponential function in (3). This is why we assume in the condition (d) that both $\delta(\pm\infty)$ are finite.

[8] If these are the finite limits, the theorem becomes that of Phragmèn and Lindelöf; cf. E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 176. The term *infinite limit* is often used in mathematics to mean that the inverse has a zero limit, regardless of how the function behaves in this limit. It was meant in reference 4 that $f(z)$ *approaches definite complex numbers which, however, do not have to be finite.* These comments are due to M. Froissart (private communication).

constant. This modification does not affect the conditions (a), (b), (c), and also makes the condition (d) satisfied. Moreover, if we prove the above theorem for this modified function, we then prove the theorem for the original function.

In fact, we can show that the above theorem is valid as long as $f(z)$ satisfies the conditions (a), (b), (c), and (d), without any further assumption regarding the behavior of $f(z)$ at the infinite end of the second cut. We should remark that the conditions are now weaker than those assumed in reference 4.

The proof of the above version of the theorem is quite simple. Since $f(z)$ satisfies the conditions (a), (b), (c), and (d), $f(z)$ has the phase representation (3). Suppose $f(z)$ approaches finite limits $f(+\infty \pm i\epsilon)$ as $z \to +\infty \pm i\epsilon$. We may assume without loss of generality that the limits $f(+\infty \pm i\epsilon)$ are not zero. According to the argument in Sec. 2, this means that the integral (7) converges and also the sum of the powers must vanish in the asymptotic form (9) of the phase representation (3). It is then a simple matter to check that the asymptotic form (9) is independent of the phase of z, but depends only on whether z goes to infinity in the upper or lower half-plane. This completes the proof of the above theorem.

4. HIGH-ENERGY BEHAVIOR OF SCATTERING AMPLITUDE

Consider an elastic scattering amplitude $A(s,t)$ as a function of s, the invariant total-energy squared, and t, the invariant momentum-transfer squared. According to Mandelstam, $A(s,t)$ is analytic in s, in a region of t near $t=0$, in the sense that the conditions (a), (b), and (c) of Sec. 2 are satisfied. The optical theorem implies that the condition (d) is also satisfied for $A(s,t)$ in a region of t near $t=0$. Thus, $A(s,t)$ has a phase representation (3). The asymptotic form (9) must, therefore, be the high-energy behavior of $A(s,t)$ in a region of t near $t=0$. The details of pion-pion and pion-nucleon scattering amplitudes have already been published.[3]

We add a few remarks. Our derivation of this power form of s for $A(s,t)$ when $s \to \infty$, strongly suggests that the power behavior is actually the case in all elastic scattering, since the condition that the integral (7) converges is sufficiently weak. An interesting point here is that the power behavior itself does not necessarily imply any specific assumptions in the complex angular-momentum plane, but is more or less a direct consequence of the usual analyticity assumption. Another remark is that the dependence of the power of s on t in this power behavior is a separate question which requires further investigation,[9] because we have exploited in our derivation the analyticity of $A(s,t)$ in s, but not the one in t.

[9] This question has been extensively discussed by Y. Nambu and M. Sugawara (to be published), which includes the two-dimensional generalization of the phase representation (3).

5. PARTIAL-WAVE AMPLITUDES

Suppose $f(z)$ is a partial-wave amplitude. Then the phase $\delta(x)$ along the physical cut is by definition the real part of the physical phase shift of this partial-wave amplitude. This is why the phase representaion (3) has been discussed[1] and used[2] by many authors in connection with partial-wave amplitudes.

We note here that the N/D representation is nothing but the phase representation (3) in which the exponential function is split into two factors, one involving the integral along the physical cut and the other including the other integrals. The inverse of the former factor is the D function, while the latter is the N function, aside from a trivial ambiguity in assigning the polynomials in (3) to D or N.

We add the following remarks. First, the N/D representation is always possible as long as $f(z)$ satisfies the conditions (a), (b), (c), and (d) of Sec. 2. Conversely, the phase representation (3) exists when the N/D representation exists because the latter requires that $\delta(\pm\infty)$ are finite individually. Secondly, the N/D representation is unique except for finite polynomials, whose orders are also uniquely defined if the numbers of zeros and poles are given. Finally, both D and N functions satisfy dispersion relations the numbers of subtractions being uniquely determined by $\delta(\pm\infty)$ and the numbers of zeros and poles. In other words, the zeros of these functions can be determined by $\delta(\pm\infty)$ and the dispersion relations which include the information regarding the subtractions and the poles. The details of how to determine the zeros of the D function and also of the inverse function are discussed in reference 5.

6. ZEROS OF FORWARD PION-NUCLEON SCATTERING AMPLITUDES

The phase representation is used in this section to locate all the zeros of the forward pion-nucleon scattering amplitudes. This is possible because the asymptotic form (9) implies that n is determined if we know $\delta(\pm\infty)$ and the dispersion relation which involves m and the over-all asymptotic behavior.

Let $T_\pm(\omega)$ be the $+$ or $-$ combination of the forward $p\pi^-$ and $p\pi^+$ amplitudes as functions of ω, the lab incident pion energy. We normalize $T_\pm(\omega)$ in such a way that

$$\mathrm{Im}T_\pm(\omega) = (q/2)[\sigma_{p\pi^-}(\omega) \pm \sigma_{p\pi^+}(\omega)], \qquad (11)$$

where $\sigma_{p\pi^\mp}(\omega)$ are the total cross sections for $p\pi^\mp$ collisions and q is the lab incident pion momentum. It then follows that

$$T_+(\mu) = 4\pi[1+(\mu/M)](a_1+2a_3)/3, \qquad (12)$$
$$T_-(\mu) = 4\pi[1+(\mu/M)](a_1-a_3)/3,$$

where M and μ are the nucleon and pion masses, respectively, and a_1 and a_3 are the S-wave scattering lengths in the channels with the total isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

It is known that $T_\pm(\omega)$ are analytic in ω in the sense that the conditions (a), (b), and (c) of Sec. 2 are satisfied. The singularities are the two cuts, $\pm\mu$ to $\pm\infty$, and the two poles at $\pm\mu^2/2M$. Crossing symmetry $T_\pm^*(\omega) = \pm T_\pm(-\omega)$, expressed for real ω approached from above the real axis, implies that the phases satisfy $\delta_\pm(\omega) = -\delta_\pm(-\omega)$. Because of the optical theorem (11), $\delta_+(\omega)$ is bounded between 0 and π in magnitude, but $\delta_-(\omega)$ does not have to be bounded because $\delta_-(\omega)$ is allowed to increase its magnitude whenever $\sigma_{p\pi^-}(\omega) - \sigma_{p\pi^+}(\omega)$ vanishes. In fact, this difference of the cross sections is known to vanish 4 times up to the highest available energy. However, it would be extremely unlikely that this difference vanishes an infinite number of times as $\omega \to \infty$. As a matter of fact, the experimental data suggest strongly that the difference in question no longer vanishes beyond the highest presently available energy. Therefore, we assume that $\delta_-(\omega)$ is also bounded at $\omega = \infty$. This means that the condition (d) is satisfied for both $\delta_\pm(\omega)$ and, therefore, both $T_\pm(\omega)$ have the phase representation (3).

The limit $\delta_+(\infty)$ is $\pm\pi/2$ if we assume that $T_+(\omega)$ becomes pure imaginary as $\omega \to \infty$. The sign $\pm$ depends upon the sign of $T_+(\mu)$ and, therefore, of a_1+2a_3[10] ac-according to (12). The limit $\delta_-(\infty)$ may be inferred from the numerical work of Höhler and Ebel.[11] They computed $\mathrm{Re}T_-(\omega)$ using the unsubtracted dispersion relation for $T_-(\omega)$ and then plotted the complex vector $T_-(\omega)$ in the $\mathrm{Im}T_-(\omega)$ versus $\mathrm{Re}T_-(\omega)$ plane as a function of ω up to 1.8 BeV. Their drawing clearly indicates that $\delta_-(\omega)$ keeps on increasing whenever $\mathrm{Im}T_-(\omega)$ vanishes. This means that $\delta_-(\infty)$ should lie between 4π and 5π, provided that $\mathrm{Im}T_-(\omega)$ no longer vanishes beyond $\omega = 1.8$ BeV. Their drawing, in fact, suggests that $\delta_-(\infty) = 4\pi + \pi/2$.

It is by now established that $T_+(\omega)$ satisfies a once-subtracted dispersion relation, while $T_-(\omega)$ an unsubtracted one. These dispersion relations are consistent with the boundary conditions[4]

$$T_+(\omega)/\omega,\ T_-(\omega) \xrightarrow[\omega\to\infty]{} \text{finite limits.} \tag{13}$$

These boundary conditions should now be compared with the asymptotic form (9). By setting the corresponding powers equal, we obtain, since $m=2$,

$$n_+ = 3 + 2\delta_+(\infty)/\pi. \tag{14}$$
$$n_- = 2 + 2\delta_-(\infty)/\pi,$$

where n_+ and n_- are the total numbers of zeros of $T_\pm(\omega)$. We should add here that n_+ and n_-, are respectively, an even and odd integer, corresponding to even and odd symmetry of $T_\pm(\omega)$.

It follows from (14) that $n_+ = 2$ when $\delta_+(\infty) = -\pi/2$ and $n_+ = 4$ when $\delta_+(\infty) = \pi/2$. Since n_- must be an odd integer, (14) now requires that $\delta_-(\infty) = 4\pi + \pi/2$ when $\delta_-(\infty)$ is somewhere between 4π and 5π. Then (14) determines n_- to be 11.

A simple numerical check on the once-subtracted dispersion relation indicates that $T_+(0) > 0$. It then follows that $T_+(\omega)$ must have at least two zeros on the imaginary axis and at least two more zeros on the gap if $T_+(\mu) \geq 0$. In summary, $T_+(\omega)$ has two zeros on the imaginary axis if $a_1 + 2a_3 < 0$, it has two zeros on the imaginary axis and two more zeros on the gap if $a_1 + 2a_3 \geq 0$.[12]

We now consider the 11 zeros of $T_-(\omega)$. One of them occurs at $\omega = 0$ because of the odd symmetry of $T_-(\omega)$. Since the complex zeros must occur in pairs of 4, there could be up to two sets of complex zeros and at least one pair of imaginary zeros. Since we have found the total number of zeros, it is now possible to find their locations from the unsubtracted dispersion relation.

Our final remark concerns the significance of the limit $\delta_-(\infty) = 4\pi + \pi/2$. This means that $T_-(\omega)$ also becomes pure imaginary in the limit of infinite energy. This is not at all trivial because $T_-(\omega)$ is the charge exchange scattering amplitude. Since both finite limits in (13) must be pure imaginary because of crossing symmetry of $T_\pm(\omega)^4$, it is now likely that the finite limits in (13) are nonzero limits. This means that $\sigma_{p\pi^\pm}(\omega)$ approach nonzero finite limits as $\omega \to \infty$, and also that $\sigma_{p\pi^-}(\omega) - \sigma_{p\pi^+}(\omega)$ approaches zero like $1/\omega$ as $\omega \to \infty$. The above behavior of this difference is not at all trivial.

ACKNOWLEDGMENTS

We wish to thank Dr. M. Froissart for valuable discussions and substantial contribution to the proof presented in Sec. 3 of this paper. We also thank Dr. M. A. Ruderman for pointing out the interesting example quoted in reference 6.

[10] $(a_1+2a_3)/3$ is $+0.0013$, according to J. Hamilton and W. S. Woolcock, Phys. Rev. **118**, 291 (1960), but -0.0027 ± 0.0023, according to W. S. Woolcock, in *Proceedings of the Aix-en-Provence International Conference on Elementary Particles, 1961* (Centre d'Etudes Nucléaires de Saclay, Seine et Oise, 1961), Vol. 1, p. 459.

[11] G. Höhler and G. Ebel (unpublished), Institute for Theoretical Nuclear Physics, Institute of Technology Karlsruhe, Germany (1962).

[12] Using only the once-subtracted dispersion relation for $T_+(\omega)$, S. Aramaki, Progr. Theoret. Phys. (Kyoto) **28**, 479 (1962), has reached the same conclusions, including that there can be no complex zeros.